11-4 How to Integrate Powers of Trig Functions

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Ex1. $\int \sin^2 \theta d\theta = \int (\sin\theta)^2 d\theta$

$\int \sin^n \theta d\theta$

n is even

$\int \frac{1-\cos(2\theta)}{2} d\theta$

$\frac{1}{2}\int 1 - \cos(2\theta) d\theta$

$\frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C$

$\boxed{\frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C}$

TRIG IDENTITIES

$\cos(2\theta) = 1 - 2\boxed{\sin^2\theta}$

$\frac{\cos(2\theta) - 1}{-2} = \frac{-2\sin^2\theta}{-2}$

$\boxed{\frac{1-\cos(2\theta)}{2} = \sin^2\theta}$

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$\int \sin^n \theta d\theta$

n is odd

Ex2. $\int \sin^5 \theta d\theta$

$\int \sin^{4}\theta \cdot \underbrace{\sin\theta \, d\theta}$ even

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$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\int (\sin^2\theta)^2 \sin\theta \, d\theta$

$\int (1-\cos^2\theta)^2 \sin\theta \, d\theta$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta \qquad d\theta = \frac{du}{-\sin\theta}$

$\int (1-u^2)^2 \cancel{\sin\theta} \cdot \frac{du}{-\cancel{\sin\theta}}$

$-\int (1-u^2)^2 du \qquad (1-u^2)(1-u^2)$

$-\int (1-2u^2+u^4) du$

$-\left(u - \frac{2}{3}u^3 + \frac{u^5}{5} + C\right) \qquad u=\cos\theta$

$-\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

Ex3. $\int \cos^{\boxed{3}}\theta \sin^4\theta\, d\theta$

$$\int \cos^m\theta \sin^n\theta\, d\theta$$

n is odd or m is odd

or both are odd

$$\int \cos^2\theta \sin^4\theta \underbrace{\cos\theta\, d\theta}_{du}$$

$$\int (1-\sin^2\theta)\sin^4\theta \cdot \underbrace{\cos\theta\, d\theta}_{du}$$

$$\int (1-u^2)u^4 \cancel{\cos\theta} \frac{du}{\cancel{\cos\theta}}$$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$$\int u^4 - u^6\, du$$

$$\frac{u^5}{5} - \frac{u^7}{7} + C$$

$$\boxed{\frac{\sin^5\theta}{5} - \frac{\sin^7\theta}{7} + C}$$

Ex4. $\int \cos^2\theta \sin^4\theta\, d\theta$

$\int \cos^m\theta \sin^n\theta\, d\theta$
n is even and
m is even

$$\int \cos^2\theta \cdot (\sin^2\theta)^2\, d\theta$$

$$\int \left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(2\theta)}{2}\right)^2 d\theta$$

$$\frac{1}{8}\int (1+\cos(2\theta))(1-\cos(2\theta))^2\, d\theta$$

$$\frac{1}{8}\int (1+\cos 2\theta)(1-2\cos 2\theta+(\cos 2\theta)^2)\, d\theta$$

$$\frac{1}{8}\int 1-2\cos 2\theta+\cos^2 2\theta+\cos 2\theta-2\cos^2 2\theta+\cos^3 2\theta\, d\theta$$

$$\frac{1}{8}\int 1-\cos 2\theta-\cos^2 2\theta+\cos^3 2\theta\, d\theta$$

$$\frac{1}{8}\left[\int 1-\cos(2\theta)\,d\theta - \int \cos^2(2\theta)\,d\theta + \int \cos^3(2\theta)\,d\theta\right]$$

$$\int 1\,d\theta = \theta \qquad \int \cos(2\theta)\,d\theta = \frac{1}{2}\sin(2\theta)$$

$$\int \cos^2(2\theta)\,d\theta = \int \frac{\cos(4\theta)+1}{2}\,d\theta = \frac{1}{2}\int(\cos(4\theta)+1)\,d\theta$$
$$= \frac{1}{2}\left(\frac{1}{4}\sin(4\theta)+\theta\right) = \frac{1}{8}\sin(4\theta)+\frac{1}{2}\theta$$

$$\int \cos^2(2\theta)\,d\theta = \int \frac{\cos(4\theta)+1}{2}\,d\theta = \frac{1}{2}\int(\cos(4\theta)+1)\,d\theta$$
$$= \frac{1}{2}\left(\frac{1}{4}\sin(4\theta)+\theta\right) = \frac{1}{8}\sin(4\theta)+\frac{1}{2}\theta$$

$$\int \cos^3(2\theta)\,d\theta = \int \cos^2(2\theta)\cos(2\theta)\,d\theta$$
$$= \int (1-\sin^2(2\theta))\cos(2\theta)\,d\theta$$

let $u=\sin(2\theta)$
$du = 2\cos(2\theta)\,d\theta$

$$= \frac{1}{2}\int (1-\sin^2(2\theta))\, 2\cos(2\theta)\,d\theta$$
$$= \frac{1}{2}\int (1-u^2)\,du$$
$$= \frac{1}{2}\left(u-\frac{1}{3}u^3\right) = \frac{1}{2}u-\frac{1}{6}u^3$$

$$= \frac{1}{8}\left[\theta - \frac{1}{2}\sin(2\theta) - \left(\frac{1}{8}\sin(4\theta)+\frac{1}{2}\theta\right) + \frac{1}{2}\sin(2\theta) - \frac{1}{6}\sin^3(2\theta)\right]$$

$$\frac{1}{8}\theta - \frac{1}{16}\sin(2\theta) - \frac{1}{64}\sin(4\theta) - \frac{1}{16}\theta + \frac{1}{16}\sin(2\theta) - \frac{1}{48}\sin^3(2\theta)$$

$$\boxed{\frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) - \frac{1}{48}\sin^3(2\theta) + C}$$

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$$\int \tan\theta \, d\theta$$

Ex5. $\int \tan x dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$$= \int \frac{\sin x}{\cos x} dx$$

$$\int \frac{\cancel{\sin x}}{u} \frac{du}{-\cancel{\sin x}} = -\int \frac{1}{u} du$$

$$-\ln|u| + C = -\ln|\cos x| + C$$

$$\ln|\cos x|^{-1} + C$$

$$\ln|\sec x| + C$$

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$\int \sec\theta d\theta$

Ex6. $\int \sec x dx$

$$\int \frac{\sec x+\tan x}{\sec x+\tan x} \sec x \, dx$$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \sec x + \tan x \qquad \frac{du}{dx} = \sec^2 x + \sec x \tan x$

$$\int \frac{\cancel{\sec^2 x + \sec x \tan x}}{u} \cdot \frac{du}{\cancel{\sec^2 x + \sec x \tan x}} \qquad dx = \frac{du}{\sec^2 x + \sec x \tan x}$$

$\ln|u| \qquad \ln|\sec x + \tan x| + C$

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Ex7. $\int sec^4 x dx$

$\int \sec \theta^m \, d\theta$

m is even

$\int \sec^2 x \cdot \underbrace{\sec^2 x \, dx}_{du}$

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$\tan^2 x + 1 = \sec^2 x$

$\int (1+\tan^2 x) \sec^2 x \, dx$

$u = \tan x \qquad \frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$\int (1+u^2) \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$

$\boxed{\tan x + \frac{1}{3}\tan^3 x + C}$

$\int 1+u^2 \, du$

$u + \frac{1}{3}u^3 + K$

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$\int \tan^m \theta \, d\theta$

m is even or odd

Ex8. $\int \tan^4 x \, dx$

$$\int \tan^2 x \cdot \tan^2 x \, dx$$

$$\int (\sec^2 x - 1) \cdot (\sec^2 x - 1) \, dx$$

$$\int \sec^4 x - 2\sec^2 x + 1 \, dx$$

$$\int \sec^4 x \, dx - 2\int \sec^2 x \, dx + \int 1 \, dx$$

$$\tan x + \frac{\tan^3 x}{3} - 2\tan x + x + C$$

$$\frac{\tan^3 x}{3} - \tan x + x + C$$

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Ex7. $\int sec^3 x dx$

$\int \sec^m \theta d\theta$
m is odd

$$\boxed{\begin{array}{c}\underline{\text{Int by Parts}}\\ \int u dv = uv - \int v du\end{array}}$$

$u = \sec x$
$du = \sec x \cdot \tan x$
$dv = \sec^2 x$
$v = \tan x$

$$\int \sec^2 x \cdot \sec x \, dx$$

$$\int \sec^3 x dx = \sec x \cdot \tan x - \int \tan x \cdot \sec x \tan x \, dx$$

$$\int \sec^3 x dx = \sec x \cdot \tan x - \int \tan^2 x \cdot \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x - \sec x \, dx$$

$$\int \sec^3 x dx = \sec x \tan x - \cancel{\int \sec^3 x dx} + \int \sec x dx$$

$$+ \int \sec^3 x dx \qquad\qquad + \cancel{\int \sec^3 x dx}$$

$$2\int \sec^3 x dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$= \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C$$

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Ex9. $\int \tan^5 \theta \sec^3 \theta d\theta$

$\int tan^m \theta \sec^n \theta d\theta$

m is odd

$\int \tan^4\theta \sec^2\theta \underbrace{\sec\theta \cdot \tan\theta \, d\theta}$ sec

$\int (\tan^2\theta)^2 \sec^2\theta \cdot \underbrace{\sec\theta \cdot \tan\theta \, d\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$(\sec^2\theta - 1)^2 \sec^2\theta \cdot$

$u = \sec\theta$

$\frac{du}{d\theta} = \sec\theta \tan\theta$

$\frac{du}{\sec\theta \tan\theta} = d\theta$

$\int (u^2-1)^2 u^2 \, du$

$\int (u^4 - 2u^2 + 1) u^2 \, du$

$\int u^6 - 2u^4 + u^2 \, du$

$\frac{u^7}{7} - \frac{2u^5}{5} + \frac{1}{3}u^3 + C$

$\frac{\sec^7\theta}{7} - \frac{2\sec^5\theta}{5} + \frac{\sec^3\theta}{3} + C$

$$\int \tan^m\theta \sec^n\theta d\theta$$

n is even

Ex9. $\int \tan^6\theta \sec^4\theta d\theta$

$\int \tan^6\theta(\tan^2\theta+1)\cdot \cancel{\sec^2\theta}\,d\theta$ $\tan\theta=u$

$\sec^2\theta=du$

$\int u^6(u^2+1)\,du$

$\int u^8+u^6\,du$

$\frac{\tan^9\theta}{9}+\frac{\tan^7\theta}{7}+C$

Ex10. $\int \tan^4\theta \sec^3\theta d\theta$

$\int tan^m\theta \sec^n\theta d\theta$

m is even and n is odd

$\int (\tan^2\theta)^2 \sec^3\theta \, d\theta$

$\int (\sec^2\theta - 1)^2 \sec^3\theta \, d\theta$

$\int (\sec^4\theta - 2\sec^2\theta + 1) \sec^3\theta \, d\theta$

$\int \sec^7\theta - 2\sec^5\theta + \sec^3\theta \, d\theta$

$\int \sec^7\theta \, d\theta - 2\int \sec^5\theta \, d\theta + \int \sec^3\theta \, d\theta$

$\vdots \qquad \vdots \qquad \vdots$

Homework

Integrating Powers of Trig Functions WS

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